

# Computationally proving triangulated 4-manifolds to be diffeomorphic

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## Abstract

We present new computational methods for proving diffeomorphism of triangulated 4-manifolds, including algorithms and topological software that can for the first time effectively handle the complexities that arise in dimension four and be used for large scale experiments.

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## 1 Introduction

In dimensions  $\leq 3$ , every topological manifold has a unique smooth structure up to diffeomorphism. In dimensions  $\geq 4$  this is no longer true: there are pairs of 4-manifolds which are *homeomorphic* (they represent the same topological manifold) but not *diffeomorphic* (they represent two distinct smooth manifolds) [9]. Finding such pairs is important; indeed, the only outstanding variant of the Poincaré conjecture asks whether one can find two non-diffeomorphic smooth 4-spheres [9].

We move this problem to the *piecewise linear* setting, which is better suited for computation. Here manifolds are given as *triangulations* (decompositions into simplices). Piecewise linear manifolds are in 1-to-1 correspondence with smooth manifolds for dimensions  $\leq 6$ , and so results translate between both settings; we use both languages interchangeably in this paper.

Despite this equivalence, work on non-diffeomorphic pairs is done exclusively in the smooth setting. The only example of two 4-manifold *triangulations* that are homeomorphic but not diffeomorphic follows a well-established result for the smooth setting [1]. One of the few candidates from the PL-world is a pair of triangulations of the  $K3$  surface (one of the four fundamental building blocks of simply connected 4-manifolds): the 16-vertex  $(K3)_{16}$  of Casella and Kühnel [6], and the 17-vertex  $(K3)_{17}$  of Spreer and Kühnel [12]. The smooth type of  $(K3)_{17}$  is canonical, but the smooth type of  $(K3)_{16}$  remains unknown. It is conjectured [12]:

**Conjecture 1.1.**  $(K3)_{16}$  and  $(K3)_{17}$  are diffeomorphic.

In the computational setting, proving that  $(K3)_{16}$  and  $(K3)_{17}$  are *homeomorphic* is easy, using the software `simpcomp` [7] in conjunction with Freedman’s celebrated classification of simply connected 4-manifolds [8]. Proving they are *diffeomorphic* is much harder. Our approach is based on a theorem of Pachner [11], which states that two triangulated manifolds are diffeomorphic if and only if they are related by a sequence of *bistellar moves* (local modifications).

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Bistellar moves offer significant challenges in dimensions  $\geq 4$ : “effective” sequences of moves can be extremely difficult to find. Indeed, the *number* of moves required to connect two diffeomorphic triangulations of size  $n$  must have no computable upper bound [10].

Here we describe work in progress towards resolving Conjecture 1.1, including effective heuristics and fast algorithms for manipulating 4-manifold triangulations with bistellar flips, and a tight lower bound on the size of a 1-vertex triangulation of the  $K3$  surface with over a million distinct realisations. This work has wider relevance, and forms the beginning of a larger project to explicitly construct and study “exotic” triangulated 4-manifolds.

## 2 Minimal triangulations of the $K3$ surface

For computation, we want to triangulate manifolds using few top-dimensional simplices. We therefore work with *generalised triangulations*, which are collections of abstract simplices whose facets are identified in pairs—these can allow far fewer simplices than the more rigid simplicial complexes. We also favour triangulations with just one vertex, which in lower dimensions offer significant advantages for both theory and computation.

*Proving* minimality is extremely difficult in three dimensions. In four dimensions, we solve this completely for the  $K3$  surface in the one-vertex setting:

**Proposition 2.1.** *For any triangulation of the  $K3$  surface we have  $f_4 \geq 146 - 6f_0$ , where  $f_4$  denotes the number of 4-dimensional simplices and  $f_0$  denotes the number of vertices. In the case where  $f_0 = 1$ , this bound is tight.*

We prove  $f_4 \geq 146 - 6f_0$  by combining (i) the fact that the  $K3$  surface is simply connected and has Euler characteristic  $\chi(K3) = f_0 - f_1 + f_2 - f_3 + f_4 = 24$ , with (ii) the *Dehn-Sommerville equations*  $2f_1 - 3f_2 + 4f_3 - 5f_4 = 0$  and  $2f_3 - 5f_4 = 0$ . Here each  $f_i$  denotes the number of  $i$ -faces of the triangulation.

We prove this bound is tight by reducing both  $(K3)_{16}$  and  $(K3)_{17}$  using bistellar moves to one-vertex triangulations with  $(f_0, f_1, f_2, f_3, f_4) = (1, 1, 234, 350, 140)$ . In three dimensions, such a “simplification” of triangulations is fast and effective [4], but in four dimensions it is far more difficult and requires the interaction of many different tools and heuristics.

Our approach incorporates: (i) classical greedy techniques, which reduce a triangulation as far as possible using local moves; (ii) “composite” moves that collapse edges and reduce triangulations near low-degree edges and triangles; (iii) simulated annealing [2], where we apply the inverses of reducing moves to escape local minima; (iv) breadth-first searching through the Pachner graph (or “flip graph”) [4].

We note that the *interaction* between these techniques is crucial: each technique failed to reduce  $(K3)_{16}$  and  $(K3)_{17}$  on its own. Again we contrast this with three dimensions, where these techniques are found to be highly effective even in isolation.

All computations were performed using the new 4-manifold toolkit in the software package *Regina* [5].

## 3 Connecting triangulations $(K3)_{16}$ and $(K3)_{17}$

To prove Conjecture 1.1 we must find a sequence of local modifications connecting  $(K3)_{16}$  and  $(K3)_{17}$ . In three dimensions, the following approach is often successful: (i) simplify both triangulations as far as possible, and then (ii) repeatedly apply random local modifications that preserve the number of simplices until both triangulations are identical. This often succeeds (provided

both triangulations represent the same manifold) because many manifolds appear to have only few distinct minimal triangulations.

In contrast, for the  $K3$  surface the number of minimal triangulations appears to be *much* larger, and so the classical approach above does not work. Instead we use a more sophisticated method to ensure that (i) every minimal triangulation is visited only once and (ii) we can detect if a longer “detour” through larger triangulations is required. Specifically, we run a dual-source breadth-first search through the *Pachner graph*, whose nodes represent triangulations of the  $K3$  surface and whose arcs represent bistellar flips that preserve the number of simplices. The two sources are our minimal one-vertex triangulations of  $(K3)_{16}$  and  $(K3)_{17}$ .

Each time we perform a local move, we must test whether the resulting triangulation has been seen before (up to combinatorial isomorphism). For this we compute the *isomorphism signature* of the triangulation [3], a polynomial-time computable hash that uniquely identifies the isomorphism type. This reduces the comparison to a fast lookup, and the overall algorithm runs in time  $O(T \log T \cdot n^2 \log n)$ , where  $T$  is the (large) number of triangulations, and  $n$  is the (small) number of simplices in each. The search parallelises well, since the bottlenecks are the hashing and performing local moves.

Thus far, the algorithm has detected 1 738 260 distinct minimal one-vertex triangulations of the  $K3$  surface. The search is ongoing, and has neither exhausted the list of minimal triangulations nor connected  $(K3)_{16}$  with  $(K3)_{17}$ . This enormous number of minimal triangulations is both interesting and surprising, offering a stark contrast to observations from dimension three.

## 4 Conclusion and future research

Proving Conjecture 1.1 would eliminate an important candidate for a pair of homeomorphic but non-diffeomorphic simply connected 4-manifolds. Moreover, as noted earlier, this work has a wider appeal: it shows for the first time how difficult problems of diffeomorphism and “exotic structures” in 4-manifold topology can be realistically tackled using computational tools.

Future developments will include: multiple-vertex triangulations containing fewer 4-simplices; a richer set of local modifications; and distributed algorithms for use on high-performance computing facilities.

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